

# Quantum non-signalling assisted zero-error classical capacity of qubit channels

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In this paper, we explicitly evaluate the one-shot quantum non-signalling assisted zero-error classical capacities  $\mathcal{M}_0^{\text{QNS}}$  for qubit channels. In particular, we show that for nonunital qubit channels,  $\mathcal{M}_0^{\text{QNS}} = 1$ , which implies that in the one-shot setting, nonunital qubit channels cannot transmit any information with zero probability of error even when assisted by quantum non-signalling correlations. Furthermore, we show that for qubit channels,  $\mathcal{M}_0^{\text{QNS}}$  equals to the one-shot entanglement-assisted zero-error classical capacities. This means that for a single use of a qubit channel, quantum non-signalling correlations are not more powerful than shared entanglement.

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## I. INTRODUCTION

While the ordinary channel capacity allows errors which can be made arbitrarily small in sufficiently many channel uses, the zero-error channel capacity does not allow any error. Thus, in many situations that no error is tolerated or only a small number of channel uses are available, it is important to consider the zero-error channel capacity [1].

Additional resources as correlations between sender and receiver in a channel can be used for communication, and they may increase its channel capacity. Indeed, it has been known that shared entanglement can increase zero-error capacities of both classical channels [2, 3] and quantum channels [4]. However, we cannot send information by only using the shared entanglement without any additional classical/quantum channel. For example, the dense coding [5] and the quantum teleportation [6] use shared entanglement as a resource, but both of them need a classical/quantum channel through which information can be sent. As more general resources, classical and quantum non-signalling (QNS) correlations have been introduced in Ref. [7] and Ref. [8], respectively. In particular, QNS correlations can be viewed as bipartite quantum channels from  $A_1 \otimes B_1$  to  $A_2 \otimes B_2$  through which the parties  $A$  and  $B$  cannot send any information to each other. Thus, it is said to be quantum non-signalling correlations, and shared entanglement and classical non-signalling correlations can be regarded as special cases of QNS correlations.

We here take into account the QNS assisted zero-error classical capacity of qubit channels. For qubit channels, the one-shot zero-error classical capacity cannot be superactivated [9] (in fact, it also holds for qutrit channels [10]), although the superactivation of quantum channels is a peculiar quantum effect with no classical analogue [4, 11]. In addition, regularization is not necessary for the (asymptotic) zero-error classical capacity of

qubit channels and even for the entanglement-assisted zero-error classical capacity [12]. From these properties, we may think that qubit channels have somewhat different properties from higher dimensional cases.

In this paper, we evaluate explicitly the one-shot QNS assisted zero-error classical capacities  $\mathcal{M}_0^{\text{QNS}}$  for qubit channels. For unital qubit channels, it is not so hard to calculate  $\mathcal{M}_0^{\text{QNS}}$  unlike nonunital case. Here, we show that  $\mathcal{M}_0^{\text{QNS}} = 1$  for any nonunital qubit channel. In other words, nonunital qubit channels cannot transmit any information with zero probability of error when assisted by entanglement, or even when assisted by QNS correlations in the one-shot setting. Moreover, we show that for qubit channels,  $\mathcal{M}_0^{\text{QNS}}$  equals to the one-shot entanglement-assisted zero-error classical capacity  $\mathcal{M}_0^{\text{SE}}$ . This means that QNS correlations are not more powerful than shared entanglement for a single use of a qubit channel.

This paper is organized as follows. In Section II, we evaluate the one-shot QNS assisted zero-error classical capacities for qubit channels. In Section III, we show that for qubit channels,  $\mathcal{M}_0^{\text{QNS}}$  equals to  $\mathcal{M}_0^{\text{SE}}$ . Finally, we summarize our results and conclude with discussion in Section IV.

## II. THE ONE-SHOT QNS ASSISTED ZERO-ERROR CLASSICAL CAPACITIES OF QUBIT CHANNELS

In this section, we explicitly calculate the one-shot QNS assisted zero-error classical capacities of Pauli channels and nonunital qubit channels using the formula in Ref. [8]. Since any unital qubit channel is unitarily equivalent to a Pauli channel [13], it is sufficient to consider Pauli channels rather than all unital qubit channels.

It is known [8] that one can calculate the one-shot QNS assisted zero-error classical capacities  $\mathcal{M}_0^{\text{QNS}}$  by some semi-definite programming (SDP) as follows.

**Lemma 1.** *For a given quantum channel  $\mathcal{N}$  from a quantum system  $A$  to a quantum system  $B$ , let  $\Upsilon(\mathcal{N})$  be the*

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quantity obtained from the following SDP:

$$\Upsilon(\mathcal{N}) = \max \text{Tr } S_A, \quad (1)$$

where the maximum is taken over all  $S_A$ 's satisfying

$$0 \leq U_{AB} \leq S_A \otimes I_B, \quad (2)$$

$$\text{Tr}_A U_{AB} = I_B, \quad (3)$$

$$\text{Tr } P_{AB}(S_A \otimes I_B - U_{AB}) = 0. \quad (4)$$

Here,  $P_{AB}$  is the projection onto the support of the Choi-Jamiołkowski (CJ) matrix  $(\mathcal{I} \otimes \mathcal{N}) |\Phi\rangle_{AB} \langle \Phi|$  of  $\mathcal{N}$ , where  $|\Phi\rangle_{AB} = \sum_k |k\rangle_A |k\rangle_B$  be the unnormalized maximally entangled state. Then  $\mathcal{M}_0^{\text{QNS}}(\mathcal{N})$  is the integer part  $\lfloor \Upsilon(\mathcal{N}) \rfloor$  of the quantity  $\Upsilon(\mathcal{N})$ .

We note that since  $\Upsilon(\mathcal{N})$  essentially depends on the projection  $P_{AB}$ ,  $\Upsilon(\mathcal{N})$  can be also denoted by  $\Upsilon(P_{AB})$ .

### A. Pauli channels

We first consider Pauli channels defined as follows.

**Definition 2.** For a probability distribution  $\{p_{ij}\}$ , the Pauli channel  $\mathcal{N}^P$  is defined by

$$\mathcal{N}^P(\rho) \equiv \sum_{i,j=0}^1 p_{ij} X^i Z^j \rho (X^i Z^j)^\dagger, \quad (5)$$

where  $X \equiv \sum_{j=0}^1 |j \oplus 1\rangle \langle j|$  and  $Z \equiv \sum_{j=0}^1 (-1)^j |j\rangle \langle j|$ .

**Lemma 3.** Assume that both  $A$  and  $B$  are two-dimensional quantum systems, and let  $P_{AB}$  be the projection defined as

$$P_{AB} = \sum_{i=1}^k |\phi_i\rangle_{AB} \langle \phi_i|, \quad (6)$$

where  $|\phi_i\rangle$ 's are mutually orthogonal and maximally entangled states. Then  $\Upsilon(P_{AB}) = 4/k$ .

*Proof.* For any  $S_A$  and  $U_{AB}$  satisfying the constraints (2), (3), and (4),

$$k \text{Tr } S_A = 2 \text{Tr } P_{AB} U_{AB} \leq 2 \text{Tr } U_{AB} = 2 \text{Tr } I_B = 4.$$

Thus we can obtain the inequality  $\text{Tr } S_A \leq 4/k$ . We now take  $(2/k)I_A$  as  $S_A$  and  $(2/k)P_{AB}$  as  $U_{AB}$ . Then it can be readily seen that the constraints (2), (3), and (4) hold, and  $\text{Tr}((2/k)I_A) = 4/k$ . Hence,  $\Upsilon(P_{AB}) = 4/k$ .  $\square$

Using the above lemma, we can obtain explicit values of  $\Upsilon$  for Pauli channels.

**Theorem 4.** Let  $\mathcal{N}^P$  be the Pauli channel with a probability distribution  $\{p_{ij}\}$ . Then  $\Upsilon(\mathcal{N}^P) = 4/k$ , where  $k$  is the number of nonzero probabilities  $p_{ij}$ .

*Proof.* Since the CJ matrix of  $\mathcal{N}^P$  is

$$\sum_{i,j=0}^1 p_{ij} (I \otimes X^i Z^j) |\Phi\rangle \langle \Phi| (I \otimes X^i Z^j)^\dagger,$$

and  $(I \otimes X^i Z^j) |\Phi\rangle$ 's are mutually orthogonal and (unnormalized) maximally entangled states, the projection  $P_{AB}$  associated with the channel  $\mathcal{N}^P$  is the sum of  $k$  mutually orthogonal and maximally entangled states as one-dimensional projections which is of the form in Eq. (6). Thus by Lemma 3, it is clear that

$$\Upsilon(\mathcal{N}^P) = \Upsilon(P_{AB}) = 4/k. \quad \square$$

The (asymptotic) QNS assisted zero-error classical capacity  $\mathcal{C}_0^{\text{QNS}}(\mathcal{N})$  of a quantum channel  $\mathcal{N}$  is defined by

$$\mathcal{C}_0^{\text{QNS}}(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Upsilon(\mathcal{N}^{\otimes n}). \quad (7)$$

Since it is not hard to show that  $\log \Upsilon$  is additive for Pauli channels, we directly have the following corollary.

**Corollary 5.** For Pauli channels  $\mathcal{N}^P$ ,  $\mathcal{C}_0^{\text{QNS}}$  is additive, and  $\mathcal{C}_0^{\text{QNS}}(\mathcal{N}^P) = \log(4/k)$ , where  $k$  is the number of nonzero probabilities  $p_{ij}$ .

**Remark 6.** The arguments in this subsection can be applied to higher dimensional Pauli channels as well. Then we can also calculate the QNS assisted zero-error classical capacities for generalized Pauli channels.

### B. Nonunital qubit channels

We now show that the value of  $\Upsilon$  is one for nonunital qubit channels. This means that in the one-shot setting, nonunital qubit channels cannot send any message with zero probability of error even when assisted by QNS correlations.

**Theorem 7.** For any nonunital qubit channel  $\mathcal{N}$ ,  $\Upsilon(\mathcal{N}) = 1$ .

*Proof.* Assume that  $\Upsilon(\mathcal{N}) > 1$ , then we will show that  $\mathcal{N}$  must be unital. Let  $J_{AB}$  be the CJ matrix of a qubit channel  $\mathcal{N}$ . We note that  $\text{Tr}_B J_{AB} = I_A$ , since  $\mathcal{N}$  is a quantum channel, and that  $\text{Tr}_A J_{AB} = I_B$  if and only if  $\mathcal{N}$  is unital. Moreover,  $\text{rank}(J_{AB}) < 4$ , since otherwise  $P_{AB} = I_{AB}$ , and so  $\Upsilon(P_{AB}) = 1$ .

We first suppose that  $\text{rank}(J_{AB}) = 1$ . Then  $J_{AB} = \alpha |\phi\rangle \langle \phi|$  for some  $\alpha > 0$  and some state  $|\phi\rangle$ . Since  $\text{Tr}_B J_{AB} = I_A$ ,  $|\phi\rangle$  is maximally entangled. Thus,  $\text{Tr}_A J_{AB} = I_B$ , and so  $\mathcal{N}$  is unital.

We now suppose that  $\text{rank}(J_{AB}) = 3$ . Then  $P_{AB} = I_{AB} - |\psi\rangle \langle \psi|$  for some state  $|\psi\rangle$ . Since  $\Upsilon(P_{AB}) > 1$ , there exist  $S_A$  with  $\text{Tr } S_A > 1$  and  $U_{AB}$  satisfying the constraints (2), (3), and (4). Then  $S_A \otimes I_B - U_{AB} =$

$\alpha |\psi\rangle\langle\psi|$  for some  $\alpha > 0$ . By tracing out the system A, we can see that  $|\psi\rangle$  must be maximally entangled. Up to local unitary, we may assume that  $P_{AB} = I_{AB} - |\phi^+\rangle\langle\phi^+|$  and  $J_{AB} = \sum_{i=1}^3 \alpha_i |\phi_i\rangle\langle\phi_i|$ , where  $\sum_i \alpha_i = 2$  with  $\alpha_i > 0$ ,  $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ , and  $|\phi_i\rangle$  are orthogonal to  $|\phi^+\rangle$ . Since  $\text{Tr}_A |\psi\rangle\langle\psi| + (\text{Tr}_B |\psi\rangle\langle\psi|)^T = I$  for any state  $|\psi\rangle$  orthogonal to  $|\phi^+\rangle$ ,  $\text{Tr}_A J_{AB} = 2I - (\text{Tr}_B J_{AB})^T = I$ . Thus,  $\mathcal{N}$  is unital.

Let us assume that  $\text{rank}(J_{AB}) = 2$  in the rest of this proof. Without loss of generality, let

$$J_{AB} = r |\psi_1\rangle\langle\psi_1| + (2-r) |\psi_2\rangle\langle\psi_2|,$$

where  $0 < r \leq 1$ , and  $|\psi_1\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$  and  $|\psi_2\rangle = \sum_{i,j=0}^1 a_{ij} |ij\rangle$  are orthogonal states for  $0 \leq \lambda \leq 1/2$ . Since  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are orthogonal,

$$a_{11} = -\sqrt{\frac{\lambda}{1-\lambda}} a_{00}. \quad (8)$$

Since  $\text{Tr}_B(J_{AB}) = I_A$ ,

$$a_{00}a_{10}^* + a_{01}a_{11}^* = 0 \quad (9)$$

and

$$r(1-2\lambda) = (2-r)(2|a_{00}|^2 + 2|a_{01}|^2 - 1). \quad (10)$$

Since  $0 < r \leq 1$  and  $\lambda \leq 1/2$ , it follows from Eq. (10) that

$$|a_{00}|^2 + |a_{01}|^2 \geq 1/2. \quad (11)$$

We divide the remaining proof into three cases according to the values of  $a_{01}$  and  $a_{00}$ ; (Case 1)  $a_{01} = 0$ , (Case 2)  $a_{00} = 0$ , and (Case 3)  $a_{00} \neq 0$  and  $a_{01} \neq 0$ .

(Case 1) From Eqs. (9) and (11),  $a_{10} = 0$ . Then  $|\psi_2\rangle = a_{00}|00\rangle + a_{11}|11\rangle$ , and  $\text{Tr}_A J_{AB} = \text{Tr}_B J_{AB} = I$ .

(Case 2) By Eq. (8),  $a_{11} = 0$ . Then we can choose an orthonormal basis  $\{|\psi_3\rangle, |\psi_4\rangle\}$  for the subspace orthogonal to  $P_{AB}$ , where  $|\psi_3\rangle = \sqrt{1-\lambda}|00\rangle - \sqrt{\lambda}|11\rangle$  and  $|\psi_4\rangle = a_{10}^*|01\rangle - a_{01}^*|10\rangle$ . Let  $S_A$  and  $U_{AB}$  satisfy the constraints (2), (3), and (4). By the constraints (2) and (4),

$$S_A \otimes I_B - U_{AB} = a |\psi_3\rangle\langle\psi_3| + b |\psi_4\rangle\langle\psi_4| + c |\psi_3\rangle\langle\psi_4| + c^* |\psi_4\rangle\langle\psi_3| \quad (12)$$

for some  $a, b \geq 0$  and  $c \in \mathbb{C}$ . Tracing out the system A on both sides of Eq. (12), by the constraint (3), we can obtain  $a(1-2\lambda) = b(1-2|a_{01}|^2)$  and  $|a_{01}|^2 \leq 1/2$ . By Eq. (11),  $|a_{01}|^2 = 1/2$ , and so  $|\psi_2\rangle$  is maximally entangled. Moreover, from Eq. (10), we can see that  $|\psi_1\rangle$  is also maximally entangled. Thus,  $\text{Tr}_A J_{AB} = I_B$ .

(Case 3) Since  $\Upsilon(\mathcal{N}) > 1$ , there are  $S_A$  with  $\text{Tr } S_A > 1$  and  $U_{AB}$  satisfying the constraints (2), (3), and (4). By the constraints (2) and (4),

$$S_A \otimes I_B - U_{AB} = \alpha |\psi_3\rangle\langle\psi_3| + \beta |\psi_4\rangle\langle\psi_4|, \quad (13)$$

where  $\{|\psi_3\rangle, |\psi_4\rangle\}$  is an orthonormal basis for the subspace orthogonal to  $P_{AB}$  and  $\alpha, \beta \geq 0$ . Let  $|\psi_3\rangle = \sum_{i,j=0}^1 b_{ij} |ij\rangle$  and  $|\psi_4\rangle = \sum_{i,j=0}^1 c_{ij} |ij\rangle$ . Since  $|\psi_3\rangle$  is orthogonal to  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , we can note that

$$b_{11} = -\sqrt{\frac{\lambda}{1-\lambda}} b_{00}, \quad (14)$$

$$a_{01}^* b_{01} = -\frac{a_{00}^* b_{00}}{1-\lambda} - a_{10}^* b_{10}. \quad (15)$$

Similar equations hold for the coefficients of  $|\psi_4\rangle$ .

Tracing out the system A on both sides of Eq. (13), by the constraint (3), the following three equalities can be obtained.

$$\text{Tr } S_A - 1 = \alpha(|b_{00}|^2 + |b_{10}|^2) + \beta(|c_{00}|^2 + |c_{10}|^2), \quad (16)$$

$$\text{Tr } S_A - 1 = \alpha \left( \frac{\lambda|b_{00}|^2}{1-\lambda} + |b_{01}|^2 \right) + \beta \left( \frac{\lambda|c_{00}|^2}{1-\lambda} + |c_{01}|^2 \right), \quad (17)$$

$$0 = \alpha \left( b_{00}b_{01}^* - \sqrt{\frac{\lambda}{1-\lambda}} b_{00}^* b_{10} \right) + \beta \left( c_{00}c_{01}^* - \sqrt{\frac{\lambda}{1-\lambda}} c_{00}^* c_{10} \right). \quad (18)$$

Multiply both sides of Eq. (16), Eq. (17), and Eq. (18) by  $-\lambda|a_{01}|^2$ ,  $(1-\lambda)|a_{01}|^2$ , and  $a_{00}^* a_{01}$ , respectively. Then we can obtain from Eqs. (8), (9), (14), and (15) that their sum becomes

$$(1-2\lambda)|a_{01}|^2(\text{Tr } S_A - 1) = 0.$$

Thus,  $\lambda = 1/2$ , and so  $|\psi_1\rangle$  is maximally entangled. Moreover, by Eq. (9) and (10), we can see that  $|\psi_2\rangle$  is also maximally entangled. Hence,  $\text{Tr}_A J_{AB} = I_B$ , and This completes the proof.  $\square$

**Remark 8.** It is clear by Lemma 1 that for nonunital qubit channels,  $\mathcal{M}_0^{\text{QNS}} = 1$ , and so the one-shot (entanglement-assisted) zero-error classical capacity is also one for nonunital qubit channels.

### III. RELATIONS TO THE ENTANGLEMENT-ASSISTED ZERO-ERROR CLASSICAL CAPACITIES

In this section, we show that  $\mathcal{M}_0^{\text{QNS}}$  is equals to the one-shot entanglement-assisted zero-error classical capacity  $\mathcal{M}_0^{\text{SE}}$  for qubit channels. This says that shared entanglement is sufficient to achieve  $\mathcal{M}_0^{\text{QNS}}(\mathcal{N})$  for qubit channels  $\mathcal{N}$ .

It has been known that for a quantum channel  $\mathcal{N}$ ,  $\mathcal{M}_0^{\text{SE}}(\mathcal{N})$  depends only on its associated subspace called

the *noncommutative graph* of  $\mathcal{N}$  [12]. Precisely, the associated subspace  $S$  is defined by  $S \equiv \text{span}\{E_i^\dagger E_j\}$ , where  $E_i$  are Kraus operators of the channel  $\mathcal{N}$ . Thus, to compare  $\mathcal{M}_0^{\text{QNS}}$  with  $\mathcal{M}_0^{\text{SE}}$ , we need to reformulate the results in the previous section in terms of the noncommutative graphs.

For a quantum channel  $\mathcal{N}$  with Kraus operators  $E_i$ , define the *Kraus operator space*  $K$  of  $\mathcal{N}$  as  $K \equiv \text{span}\{E_i\}$ . We note that  $S = \text{span}\{G_j^\dagger G_k\}$  for any orthonormal basis  $\{G_j\}$  of  $K$ , and that there exists an orthonormal basis  $\{F_j\}$  for  $K$  such that the CJ matrix  $J_{AB}$  of  $\mathcal{N}$  can be expressed as  $J_{AB} = \sum_j a_j (I \otimes F_j) |\Phi\rangle \langle \Phi| (I \otimes F_j^\dagger)$ , where  $a_j > 0$  [8]. Then we obtain the following theorem.

**Theorem 9.** *For any qubit channel  $\mathcal{N}$ , let  $S$  be the noncommutative graph of  $\mathcal{N}$ . Then  $\mathcal{M}_0^{\text{QNS}}(\mathcal{N}) = 4/\dim(S)$ .*

*Proof.* First, let us consider unital qubit channels. As stated at the beginning of the previous section, it is sufficient to consider Pauli channels. Let  $\mathcal{N}^P$  be a Pauli channel with the probability distribution  $\{p_{ij}\}$ , and let  $k$  be the number of nonzero probabilities  $p_{ij}$ . It is easy to calculate the dimension of the noncommutative graph  $S$  of  $\mathcal{N}^P$  according to  $k$ . Indeed, when  $k = 1, 2, 3$ , and 4,  $\dim(S) = 1, 2, 4$ , and 4, respectively. By Theorem 4,  $\mathcal{M}_0^{\text{QNS}}(\mathcal{N}^P) = 4/\dim(S)$ .

We now consider nonunital qubit channels. By Theorem 7, it is sufficient to show that  $\dim(S) = 4$ . Let  $J_{AB}$  be the CJ matrix of a nonunital qubit channel  $\mathcal{N}$ . Then  $\text{rank}(J_{AB}) > 1$ , since otherwise the associated channel must be unital as in the proof of Theorem 7. When  $\text{rank}(J_{AB}) = 4$ , the associated projection  $P_{AB}$  is equals to  $I$ , and so it is clear that  $\dim(S) = 4$ .

Assume that  $\text{rank}(J_{AB}) = 2$  or 3. Let

$$J_{AB} = \sum_{i=1}^3 r_i (I \otimes F_i) |\Phi\rangle \langle \Phi| (I \otimes F_i^\dagger),$$

where  $r_1, r_2 > 0$ ,  $r_3 \geq 0$ ,  $\sum_{i=1}^3 r_i = 2$ , and  $\{F_i\}$  is an orthonormal basis for the Kraus operator space  $K$  of  $\mathcal{N}$ . Without loss of generality, we can let  $F_1 = \sqrt{a}|0\rangle\langle 0| + \sqrt{1-a}|1\rangle\langle 1|$ ,  $F_2 = \sum_{s,t=0}^1 b_{st}|s\rangle\langle t|$ , and  $F_3 = \sum_{s,t=0}^1 c_{st}|s\rangle\langle t|$ , where  $1/2 \leq a \leq 1$  and  $b_{st}, c_{st} \in \mathbb{C}$ . We note that when  $r_3 = 0$ , it becomes the case of that  $\text{rank}(J_{AB}) = 2$ , and that  $\text{Tr}_A J_{AB} \neq I_B$ , since  $\mathcal{N}$  is nonunital.

From the orthonormality of  $\{F_i\}$ , we obtain the following equalities

$$0 = \sqrt{a}b_{00} + \sqrt{1-a}b_{11}, \quad (19)$$

$$0 = \sqrt{a}c_{00} + \sqrt{1-a}c_{11}, \quad (20)$$

$$0 = a(b_{01}^*c_{01} + b_{10}^*c_{10}) + b_{11}^*c_{11}, \quad (21)$$

$$a = a(|b_{01}|^2 + |b_{10}|^2) + |b_{11}|^2, \quad (22)$$

$$a = a(|c_{01}|^2 + |c_{10}|^2) + |c_{11}|^2. \quad (23)$$

Since  $\mathcal{N}$  is a quantum channel,  $\text{Tr}_B J_{AB} = I_A$ , which

gives the following equalities

$$1 = r_1 a + r_2(|b_{00}|^2 + |b_{10}|^2) + r_3(|c_{00}|^2 + |c_{10}|^2), \quad (24)$$

$$0 = r_2(b_{00}^*b_{01} + b_{10}^*b_{11}) + r_3(c_{00}^*c_{01} + c_{10}^*c_{11}). \quad (25)$$

First, let us consider when  $a = 1/2$ . By Eqs. (19) and (20),  $b_{00} = -b_{11}$  and  $c_{00} = -c_{11}$ . Then  $\text{Tr}_A J_{AB} = \text{Tr}_B J_{AB} = I$ , hence this is a contradiction to being nonunital.

Let us now assume that  $a \neq 1/2$  and  $\text{rank}(J_{AB}) = 3$ , that is,  $r_3 > 0$ . We consider the matrix  $M_0$  whose columns are  $|F_i^\dagger F_j\rangle$ , where  $1 \leq i, j \leq 3$  and  $|A\rangle \equiv \sum_{s,t=0}^1 a_{st}|s\rangle\langle t|$  for a matrix  $A = \sum_{s,t=0}^1 a_{st}|s\rangle\langle t|$ . Then by applying elementary operations properly on  $M_0$  we obtain the following matrix

$$M = \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & M_1 & M_2 & M_3 & & \end{array} \right), \quad (26)$$

where

$$M_1 = \begin{pmatrix} \sqrt{a}b_{01} & \sqrt{1-ab_{10}^*} \\ \sqrt{1-ab_{10}} & \sqrt{a}b_{01}^* \end{pmatrix}, \quad (27)$$

$$M_2 = \begin{pmatrix} \sqrt{a}c_{01} & \sqrt{1-ac_{10}^*} \\ \sqrt{1-ac_{10}} & \sqrt{a}c_{01}^* \end{pmatrix}, \quad (28)$$

$$M_3 = \begin{pmatrix} b_{00}^*c_{01} + b_{10}^*c_{11} & b_{01}^*c_{00} + b_{11}^*c_{10} \\ b_{01}^*c_{00} + b_{11}^*c_{10} & b_{00}^*c_{01} + b_{10}^*c_{11} \end{pmatrix}, \quad (29)$$

$$z = c_{00}^*c_{01} + c_{10}^*c_{11}. \quad (30)$$

We will show that  $\text{rank}(M) = 4$ , that is,  $\dim(S) = 4$ . To do this, we suppose that  $\text{rank}(M) < 4$ . Then we note that any two-by-two submatrix of the lower-right block of  $M$  consisting of  $M_1$ ,  $M_2$ , and  $M_3$  has zero determinant.

Assume that  $a = 1$ . By Eqs. (19) and (20),  $b_{00} = 0 = c_{00}$ . Since  $\det(M_3) = 0$ ,

$$|b_{10}|^2|c_{11}|^2 = |b_{11}|^2|c_{10}|^2. \quad (31)$$

Since  $\det(M_1) = 0$  and  $\det(M_2) = 0$ , from Eqs. (21), (22), and (23),

$$|b_{10}|^2|c_{10}|^2 = |b_{11}|^2|c_{11}|^2, \quad (32)$$

$$|b_{10}|^2 + |b_{11}|^2 = |c_{10}|^2 + |c_{11}|^2 = 1. \quad (33)$$

It is straightforward from Eqs. (31), (32), and (33) to obtain the following equalities

$$|b_{10}|^2 = 1/2 = |c_{10}|^2. \quad (34)$$

Then, from Eq. (24),  $r_1 = 0$ . This is a contradiction, and hence  $\text{rank}(M) = 4$ .

Assume that  $1/2 < a < 1$ . Since  $\det(M_1) = 0$  and  $\det(M_2) = 0$ ,

$$|b_{01}|^2 = \frac{1-a}{a}|b_{10}|^2, \quad (35)$$

$$|c_{01}|^2 = \frac{1-a}{a}|c_{10}|^2. \quad (36)$$

Then, from (22) and (23),

$$|b_{10}|^2 + |b_{11}|^2 = a = |c_{10}|^2 + |c_{11}|^2. \quad (37)$$

$$\text{Since } \det \begin{pmatrix} \sqrt{ab_{01}} & \sqrt{ac_{01}} \\ \sqrt{1-ab_{10}} & \sqrt{1-ac_{10}} \end{pmatrix} = 0,$$

$$b_{01}c_{10} - b_{10}c_{01} = 0, \quad (38)$$

$$\text{and since } \det \begin{pmatrix} \sqrt{ab_{01}} & \sqrt{1-ac_{10}^*} \\ \sqrt{1-ab_{10}} & \sqrt{ac_{01}^*} \end{pmatrix} = 0, \text{ from Eq. (21),}$$

$$b_{10}c_{10}^* + b_{11}c_{11}^* = 0. \quad (39)$$

$$\text{Since } \det \begin{pmatrix} \sqrt{ab_{01}} & b_{00}^*c_{01} + b_{10}^*c_{11} \\ \sqrt{1-ab_{10}} & b_{01}^*c_{00} + b_{11}^*c_{10} \end{pmatrix} = 0, \text{ by Eqs. (19), (20), (35), and (38),}$$

$$\sqrt{1-a}|b_{10}|^2c_{11} = \sqrt{ab_{10}}b_{11}^*c_{01}. \quad (40)$$

Then, from Eqs. (36) and (37), we obtain

$$|b_{10}|(|b_{10}| - |c_{10}|) = 0. \quad (41)$$

Thus,  $b_{10} = 0$  or  $|b_{10}| = |c_{10}|$ .

Suppose that  $b_{10} = 0$ . By Eqs. (19), (20), (37), and (39), it is not hard to show that  $|b_{00}|^2 = 1 - a$  and  $|c_{00}|^2 + |c_{10}|^2 = a$ . From Eq. (24),  $r_3 = 0$ . This is a contradiction, and so  $b_{10} \neq 0$  and  $|b_{10}| = |c_{10}|$ . By Eqs. (35), (36), (37), and (39),  $|b_{st}|^2 = a/2 = |c_{st}|^2$  for any  $s, t \in \{0, 1\}$ . From Eq. (24),  $a = 1/2$ . This is a contradiction, and so  $\text{rank}(M) = 4$ .

We now assume that  $a \neq 1/2$  and  $\text{rank}(J_{AB}) = 2$  in the rest of this proof. As in the case of that  $\text{rank}(J_{AB}) = 2$ , let us consider the matrix whose columns are  $|F_i^\dagger F_j\rangle$ , where  $1 \leq i, j \leq 2$ . Then by applying elementary operations properly we obtain the following matrix

$$N = \left( \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \sqrt{ab_{01}} & \sqrt{1-ab_{10}^*} \\ 0 & 0 & \sqrt{1-ab_{10}} & \sqrt{ab_{01}^*} \end{array} \right), \quad (42)$$

Suppose that  $\text{rank}(N) < 4$ . Then since  $\det(N) = 0$ ,

$$a|b_{01}|^2 - (1-a)|b_{10}|^2 = 0. \quad (43)$$

By Eq. (25),

$$|b_{00}|^2|b_{01}|^2 = |b_{10}|^2|b_{11}|^2. \quad (44)$$

Then from Eqs. (19), (22), and (43),

$$|b_{10}|^2(a - |b_{10}|^2) = 0. \quad (45)$$

If  $b_{10} = 0$ , by Eq. (43),  $b_{01} = 0$ , and so  $\text{Tr}_A J_{AB} = \text{Tr}_B J_{AB} = I$ . This is a contradiction, and hence  $|b_{10}|^2 = a$ . By Eqs. (19), (22), and (43),  $|b_{00}|^2 + |b_{10}|^2 = a$ . Then by Eq. (24),  $a = 1/2$ , which implies being unital. Therefore, we can conclude that  $\text{rank}(N) = 4$ , and hence  $\dim(S) = 4$ .  $\square$

Since  $\mathcal{M}_0^{\text{SE}}(\mathcal{N})$  depends on the noncommutative graph  $S$  of the channel  $\mathcal{N}$ ,  $\mathcal{M}_0^{\text{SE}}(\mathcal{N})$  can be denoted by  $\mathcal{M}_0^{\text{SE}}(S)$ . For quantum channels with qubit inputs, that is, noncommutative graphs  $S \subset \mathcal{L}(\mathbb{C}^2)$ ,  $\mathcal{M}_0^{\text{SE}}(S)$  and the (asymptotic) entanglement-assisted zero-error classical capacity  $\mathcal{C}_0^{\text{SE}}(S)$  can be obtained from Ref. [12] as in the following proposition.

**Proposition 10.** *For a noncommutative graph  $S \subset \mathcal{L}(\mathbb{C}^2)$ ,  $\mathcal{C}_0^{\text{SE}}(S) = \log \mathcal{M}_0^{\text{SE}}(S)$ . Moreover, when  $\dim(S) = 1, 2, 3$ , and 4,  $\mathcal{M}_0^{\text{SE}}(S) = 4, 2, 2$ , and 1, respectively.*

In fact, as shown in the proof of Theorem 9,  $\dim(S) \neq 3$  for qubit channels. From Theorem 9 and Proposition 10, we directly obtain the following corollary.

**Corollary 11.** *For any qubit channel  $\mathcal{N}$ ,  $\mathcal{M}_0^{\text{SE}}(S) = \mathcal{M}_0^{\text{QNS}}(\mathcal{N})$  and  $\mathcal{C}_0^{\text{SE}}(\mathcal{N}) = \log \mathcal{M}_0^{\text{QNS}}(\mathcal{N})$ .*

**Remark 12.** Corollary 11 means that QNS correlations are not more powerful than shared entanglement for a single use of a qubit channel.

#### IV. CONCLUSION

We have considered the one-shot QNS assisted zero-error classical capacities  $\mathcal{M}_0^{\text{QNS}}$  of qubit channels. First, we have calculated the exact values of  $\mathcal{M}_0^{\text{QNS}}$  for Pauli channels and nonunital qubit channels, and then we have shown that  $\mathcal{M}_0^{\text{SE}}(S) = \mathcal{M}_0^{\text{QNS}}(\mathcal{N})$  for any qubit channel.

Moreover, we can present examples of quantum channels  $\mathcal{N}$  such that  $\mathcal{C}_0^{\text{SE}}(\mathcal{N}) = 0 < \mathcal{C}_0^{\text{QNS}}(\mathcal{N})$ . (Such examples for classical channels were already known [7]). This implies that QNS correlations can make useless channels useful although the shared entanglement cannot do so. Let us consider Pauli channels with three nonzero probability  $p_j$ . Then Theorem 4 and Corollary 5 shows that  $\log \mathcal{M}_0^{\text{QNS}} = 0 < \log(4/3) = \mathcal{C}_0^{\text{QNS}}$ . By Corollary 11, we can obtain the inequality  $\mathcal{C}_0^{\text{SE}} = 0 < \mathcal{C}_0^{\text{QNS}}$ .

Another example is the (nonunital) extremal qubit channel  $\mathcal{N}(\cdot) = \sum_{j=1}^2 E_j \cdot E_j^\dagger$ , where  $E_1 = \cos \theta |0\rangle\langle 0| + \cos \varphi |1\rangle\langle 1|$ ,  $E_2 = \sin \varphi |0\rangle\langle 1| + \sin \theta |1\rangle\langle 0|$ , and  $\theta, \varphi \in \mathbb{R}$  with  $\cos^2 \theta \neq \cos^2 \varphi$  [14]. By the criterion for determining whether  $\mathcal{C}_0^{\text{QNS}} = 0$  in Ref. [8], we can see that  $\mathcal{C}_0^{\text{QNS}} > 0$ , while  $\mathcal{C}_0^{\text{SE}} = 0$  by Theorem 7 and Corollary 11.

There are many questions about  $\mathcal{C}_0^{\text{QNS}}$ . One of them is determining the value of  $\mathcal{C}_0^{\text{QNS}}$  for quantum channels. In fact, the explicit value of  $\mathcal{C}_0^{\text{QNS}}$  is yet unknown, even for the above nonunital extremal qubit channels. Another is the additivity of  $\mathcal{C}_0^{\text{QNS}}$ . For any unital qubit channel,  $\mathcal{C}_0^{\text{QNS}}$  is additive by Corollary 5. Moreover,  $\mathcal{C}_0^{\text{QNS}}$  is additive for any classical-quantum channel [8]. However, it is not known that the additivity holds for any quantum channel. Although the above questions are proved for unital qubit channels in this paper, our work could shed light on the mentioned problem.

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